XVI. *Essay on the Resolution of Algebraic Equations: attempting to distinguish particularly, the real Principle of every Method, and the true Causes of the Limitations to which it is subject. By* Giffin Wilson, *Esq. Communicated by* Edward Whitaker Gray, *M. D. Sec. R. S.*

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INTRODUCTION.

1. THE *practical* management of algebraic equations, as far as respects the solution of problems depending upon them, is well understood; but their *general theory*, being considered as an abstruse and purely speculative subject, is no where, that I have seen, so fully analysed, as with all the assistance to be derived from the application of the principles of combination, it appears to me it might be.

2. The difficulties under which the higher branches of algebra still labour are generally known. No degree of equations beyond the second, is yet perfectly resolved: cubics present frequently an irreducible case: biquadratics have, by several methods, been reduced to cubics; but no formula exhibiting to the eye the actual resolution of a biquadratic has yet appeared: and, for the fifth degree, and all upwards, not even a clue which promises a general resolution has been struck out, by the continued labour and ingenuity of mathematicians for several centuries.

3. This failure in the chain, beginning at the third degree,

and its breaking off entirely after the fourth, have been very puzzling and mortifying circumstances to the cultivators of algebra. Having in the first degrees proceeded upon apparently very general principles, and made a seeming progress towards a general resolution of equations, it is provoking to find it suddenly interrupted, not to be resumed *by any contrivance.* Various causes have been assigned for so remarkable a difficulty; but the generality of those causes, as commonly given, do not reach the principle. It has been usual for operators, when they found their methods fail, to look back till they could detect some inconsistence or impossibility in their work, and to suppose the difficulty explained, by pointing out the period at which such an error is made. The power and richness of the algebraic calculus affords numerous ways of compassing the same thing; and, as *all* of them fail when applied to this object, there is necessarily a point in *every one of them*, at *which* some inconsistence or impossibility is introduced: thence, a number of different causes may be imagined. In Dr. Waring's *Meditationes Algebraicæ*, (p. 182.) may be seen several concurrent reasons assigned, why the methods there shewn, and Dr. Waring's own, (undoubtedly the most general of any of them, since it proceeds upon one principle to the fifth degree,) cannot apply further: but, all reasons drawn from the data of any particular method, (like that commonly given for the imperfection in Cardan's Rule, which I shall examine hereafter,) though very just in themselves, cannot be conclusive: they indisputably shew, why the precise method to which they respectively apply must fail; but that does not exclude the expectation that some other, founded upon different principles, may succeed. The question therefore recurs: Is there not some paramount funda-

mental reason for this general failure? If there can be shewn to be any thing *in the nature of abstract quantity*, which governs the several orders of quantities from which equations are framed, and leads *directly* to the distinctions and limitations practice discovers, *that* will reach the difficulty at its source, and afford the satisfaction desired.

4. I think, that by turning the course of our inquiry rather to examine how we come to succeed at all, in resolving *any* degree of equations, than why our success is so limited, the *true* principle upon which their resolution must depend will appear; and with what probability, and by what means, (if possible,) we may expect to render our methods more perfect. With this idea, I shall take a concise view of the nature and resolution of equations in general; pointing out the common difficulty, and by what circumstances that difficulty is, in certain cases, lessened or removed; confining myself always to the *principle* of each step, *and a strict analysis of the result,* avoiding all detail of mere operation; and, without pretending to much novelty upon a subject already so beaten, I persuade myself, such an investigation will lead to some conclusions *which have not been remarked*, and which are both curious and important.

CHAP. I.

Of the Resolution of Equations in general.

5. Equations, in that part of algebra which treats of their general resolution, are usually considered to be reduced to *one general form,* for the greater convenience of comparing them,

i. e. to their lowest rational dimension, with unity always for the coefficient of the highest power of the unknown quantity; in which state, every simple equation is already resolved. The resolution of all other degrees, is the finding the simple equations of which they are compounded: but, to do this in a general manner, it is evident we must seek, instead of the particular equations themselves directly, a general expression representing them all; which general expression is called the formula of resolution, such as, the common quadratic resolution, or that given for cubics by CARDAN's Rule.

6. These formulæ, properly speaking, are rather the *reversion* of an equation, than the resolution of it: for, although the unknown quantity be evolved or reduced to a simple dimension, the known parts are necessarily involved or affected with a surd at least as high as the dimension of the equation, in order to exhibit the proper number of correspondent values belonging to the unknown quantity in an equation of that degree. Thus, the equation $(x^2 - px + q = o)$ and its common resolution $\left(x = \frac{p \pm \sqrt{p^2 - 4q}}{2}\right)$ *are both the same quadratic;* only, under the first form, the unknown quantity, being of the dimension of the second degree, has *two* values; whereas, in the second form, it has only *one*, and the double value is transferred, by the quadratic surd, to the known parts on the opposite side of the equation. Thus also, the equation $(x^3 - qx + r = o)$ and the CARDANIC formula belonging to it $\left(x = \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}} + \sqrt[3]{-\frac{r}{2} - \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}\right)$ are, in the same manner, *the same cubic merely reverted.* But, as equations are usually denominated from the dimension of the unknown quantity, these reso-

lutions are commonly deemed simple equations: they may in this view be defined to be, *the simple equations* that the original quadratic, cubic, or other higher given equation, *contained in power*, since they express the nature and form of a *quantity which*, by involution or reverting the operation, re-produces it; as the root of any power, being reinvolved, returns to the power from which it was extracted. This fixed and visible connection between the equation and the general formula for its roots, throws a beauty and elegance into the method of *pure algebraic resolution*, which none of the others, such as the method of divisors, and all the contrivances for approximation, can pretend to. For, when by any of those methods we have obtained one or more separate roots, the relation to the original equation is no longer perceivable; but *here* the chain is perfect. The equation leads to the resolution: the resolution embraces at once all the correspondent roots; and, when reinvolved, proves the operation, by reproducing the original equation. Thus, for example, if ($x^2 - 5x + 4 = o$), and it be perceived, or found by any conjectural method, that unity is one of the roots of that equation, there is no discernible connection between the simple equation expressing ($x = 1$) and the original equation; *no* transformation of *one* will produce the *other*. This latter equation ($x = 1$), though truly expressing a numeral root of the former, is no more a resolution of it than of the equations ($x^2 - 6x + 5 = o$), ($x^2 - 7x + 6 = o$), or any other of the infinite number of equations of which unity is a root; whereas, the algebraic resolution of ($x^2 - 5x + 4 = o$) *viz.* $\left(x = \frac{5 \pm \sqrt{25 - 16}}{2}\right)$, which equally expresses (1), and (4) the other root, needs only to be cleared of its radical, to shew itself

but another form of the same equation; and gives ($x^2 - 5x + 4 = 0$) as at first.

7. This view of the algebraic resolution of an equation shews, that it does not so much aim at giving us the roots themselves, as the basis or common principle of their artificial combination in the equation to which it applies; pointing out some form of a perfect power, of which they may be conceived to be the correspondent natural roots. From which it follows, that if the transformation required to be made in the given equation be possible, or such as can *really* be effected, the resolution will be real; for every real power has some real root: but that if, on the contrary, the power into which the equation is conceived to be transformed be merely imaginary, the resolution must be so too; for all the roots of an imaginary power are themselves imaginary. It doth not therefore depend upon the nature of the roots of the equation *themselves*, but on the form which the equation must assume to become a perfect power, to determine, whether the resolution be real or imaginary: so that the nature of the resolution, and *that of the roots* of an equation may be very different, as we know is frequently the case; particularly in the resolution of cubic equations by CARDAN'S rule, where, when the roots are real, the resolution is almost always imaginary. This has seemed to surprize and perplex some writers very much, who have treated it as at best a paradox, if not a contradiction,* but surely without cause; for, as the formula affects only to be an ideal representation of the mechanism or structure of a perfect power

* Vide PLAYFAIR on the Arith. of Impossibles, Phil. Trans. 1778, p. 318; Dr. HUTTON on Cubic Equations, ditto, 1780, page 387; and Mr. BARON MASERES, *Script. Logarith.* Vol. II. p. 456.

answering to the given affected equation, it may be expected to be clear or complicated, real or imaginary, *not* as the roots themselves are simple and real, but as the principle of their union, *of which only it is truly the index*, is near or remote: it merely shews the central point of their combination, which, like the centre of gravity, suspension, or any other power, may not actually exist in *any* of the bodies whose motions it governs, but in some imaginary point without, and remote from them all. Had the nature of the algebraic resolution of an equation been considered in this light, and the forms to which they are proposed to be reduced, been compared with the original forms of the roots in the given equation, no surprize or appearance of paradox could have arisen in the matter; but it must have been clearly perceivable, what cases would admit of *real*, and what only of *imaginary* resolutions, as will be shewn hereafter.

I have dwelt the longer upon the nature of the *algebraic resolution* of an equation, because it is a very curious subject, about which many errors and inconsistencies have been fallen into, though hardly any direct examination of it is to be found in any of our books. It is the sole method of obtaining a complete general answer to any problem. It makes algebra consistent with itself, and sufficient to solve its own difficulties, without foreign aid, (from series or other branches;) and, in *all* cases where any *general ulterior use* is to be made of the resolution of an equation, is the only method that avails at all.

8. In order to obtain this general resolution, the common methods have been, (without considering the nature of the roots,) to attempt some universal reduction in the *forms* of equations; as,

1st. The destroying their intermediate terms, and converting them into pure powers. Or,

2dly. The discovering some constant complement which will always raise them to the *nearest* perfect power. In both which cases, the resolution will afterwards be nothing more than simple extraction of the proper root. Or,

3dly. The assuming some convenient formula with indeterminate coefficients; and, by assigning their values properly, adapting it to every case.

It would be going to too great a length, to give distinct examples here, of the application of these methods. Numerous instances of *each* of them are given in the common books of algebra, which usually treat them as separate and distinct from each other; but the fact is, *they are all in truth the same.* Whoever tries them separately, will find, however variously they seem to set out, they lead precisely to the same conclusions, and fail precisely in the same points. A quadratic, whether resolved by completing the square, or by expunging the second term: a cubic, whether resolved by CARDAN's rule, or by completing the cube, or by assuming a resolution, as suggested in Dr. WARING's *Meditationes Algebraicæ*, (p. 179, 180.) present the *same* formula of resolution, and the *same* limitations and irreducible cases. And the reason is easily found. To complete the requisite power, (according to the index of the equation,) or to destroy the intermediate terms, occasions an alteration in *just the same number of terms;* it is only the particular relation they are required to bear to each other that is varied. In the *one* case, they are all to be equal, (or equal to nothing;) in the other, to correspond respectively with the

known law of the binomial theorem, which gives the unciæ of a regular power. *Both* depend upon the practicability of a more general problem, of which they are but specific cases; *viz.* the problem "*to give the coefficients of an equation any general determinate relation.*" If *that* were practicable, and it were possible to mould them *so* as to establish a general relation between them, (or any required number of them,) it is easy to perceive, that the particular relation must be a secondary consideration; and that, wherever the *same number* of terms are to be acted upon, the same means that might make them equal, might give them any other proportion at pleasure.

9. However, of all these methods, and any other of the kind, it is to be observed, that the principle is demonstrably a false assumption. For, if it be once admitted that the construction of equations, and the laws of the successive coefficients received ever since VIETA's time, be true; or that all equations are formed invariably in the same manner, from the continual multiplication of the simple equations of their roots, which experience confirms without any exception;* it follows, that the nature of the roots must infallibly govern that of the equation derived from them; that the same form of equation can *only* be produced by the same forms of roots; and therefore, before *all sorts* of equations can be made into pure or perfect powers, or be given any other general shape, it must be shewn, that *all quantities are capable of taking the forms required to produce equations of that sort*, which will presently be seen to be impossible. If those who have lost their time and labour in vain

* Some algebraists, affecting to reject the use of negative quantities, have been compelled to dispute the generally received theory of the construction of equations; but they have not been able to suggest any other.

endeavours to improve these general methods, had, instead of involving themselves in a labyrinth of substitution and process, upon the chance of some means of simplification presenting itself, considered beforehand the probability of success, the imperfection of CARDAN'S rule would never have appeared a paradox, nor the interruption of all further progress by it have given room for surprise. They must have seen, that no equation beyond a quadratic can admit of a *real* extinction of its intermediate terms. In the general equation $(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}$ &c. $= 0)$, (p) being the sum of the roots, and (q) the sum of their combinations in pairs, by Sir I. NEWTON'S theorem for finding the sums of the powers of the roots, $(p^2 - 2q)$ will be the sum of their squares; and therefore, if both (p) and (q) vanish, the sum of the squares of the roots must vanish also; which can *never* happen with real quantities. Besides this, in attempting to destroy many intermediate terms at once, we know by experience, the equations that become incidentally necessary to be solved, rise to a much higher dimension than the given equation; so that our labour, in this respect, defeats itself.

10. Nor will these difficulties be avoided, if we abandon the idea of a general resolution, and attempt to work out the roots separately: although the number of coefficients is always sufficient to afford a distinct equation to *each* root, and therefore, by the common principles of indeterminate equations, will clearly determine them all; and would also find them, if the equations afforded by the coefficients were all of the same degree; but they rise successively, and, from the drawing them together, in order to expunge the several unknown quantities, the index of the reducing equation *increases* so as to defeat the

operation. To shew this, let us recur to the general equation before given $(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} = 0)$; suppose its (n) roots to be represented by $(a, b, c, d,$ &c. n*), then, by the construction of equations, we have (n) distinct equations from the several coefficients in succession; *viz.*

$a + b + c + d$ &c. . . . . $+ n$ - - in number $(n) = p$,

$ab + ac + ad$ &c. - - - - - - - $\left(n \times \frac{n-1}{2}\right) = q$,

$abc + abd$ &c. - - - - $n \times \frac{n-1}{2} \times \frac{n-2}{3} = r$,

$(abcde$ &c. $n)$, or the product of them all, being the coefficient of the last term. Now, as we have (n) equations, and (n) indeterminate quantities, it is evident, that by employing each equation successively to determine *one* quantity, the whole will be determined. But the equations are not *all* of the *same* degree: the first, is a simple equation: the second, being composed on one side wholly of products by two, is in degree a quadratic; the third, for the same reason, a cubic; and so on. If the first of these equations be used to determine (a), we shall have $(a = p - b - c - d$ &c. $- n)$; inserting that value for (a) in the second equation, it becomes the quadratic $(pb - b^2 + pc - c^2 + pd - d^2 - bc - bd$ &c. $= q)$. If that quadratic be solved to determine (b), and the values of $(a$ and $b)$ be inserted in the third equation, it becomes the cubic $(c^3$ &c. . . $= r)$. Moreover, the quadratic having *two* roots, its solution will have

* The nature of the roots is not material in this place; whether affirmative or negative, real or imaginary, they have just the same operation in forming the coefficients of the equation. I have however *throughout* chosen, wherever I could, to give examples capable of being tried by real and affirmative roots; and, for that purpose, have uniformly made the signs of the coefficients alternately affirmative and negative.

introduced a quadratic surd. Before therefore we can proceed to employ the third equation to determine (c), it must be squared to clear it of that surd, and of course will then rise to the 6th degree. The solution of such a dimension (if admitted for the present to be equally possible) must introduce higher radicals; and, by the intrusion of these superfluous roots at every stage, our labour increases, instead of diminishing. This is the difficulty alluded to before; and, as we have appropriated already *all* our subordinate equations, we have nothing to oppose it. It therefore seems hopeless, to expect to make any general impression upon indeterminate equations, without more help, beyond the mere knowledge of the constitution of the coefficients.

11. This difficulty, however, is wholly removed by the least circumstance that discloses any particular relation amongst the coefficients of an equation, independent of the general law of their construction. This, of course, whenever it occurs, furnishes new conditions and means of comparing the terms. Every particularity in the coefficients that gives specific varieties to the forms of equations, must, from the nature of their construction, have its source in some particular relation between two or more of the roots, and therefore, as far as that relation extends, detects them infallibly. The observation of the forms and relations of the coefficients under different species of equations, and the correspondent inferences to be drawn, as to the connection of their roots, would form a curious and very useful part of a complete treatise upon the whole doctrine of equations, which is a work much wanted. The most striking of these relations will be obvious, or familiar, to the reader who has at all considered the nature of the subject; such as, that equations deficient in every alternate term arise from pairs of equal roots

with opposite signs ($\pm a, \pm b$ *&c.*); that those whose terms on both sides the middle term are alike (which are generally called recurring equations) arise from pairs of roots, of which each pair contains a quantity and its reciprocal $\left(a \frac{1}{a}, b, \frac{1}{b} \text{ \&c.}\right)$; together with MACLAURIN's demonstration of the particularities of the coefficients when an equation has equal roots.* And the extent to which these notices might easily be carried, from observation of the effects of the different sorts of proportion, and all other relations, is prodigious. But my present concern is merely with the result, supposing from any means a relation to be previously discovered affecting any number of the roots. For example,—suppose, in the above given equation, ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}$ *&c.* $= 0$), whose roots we called (a, b, c, d *&c.* $\ldots n$), we happened to know that two of the number (a and b) were equal; then, since they might *both* be expressed by the same character, the (n) roots of the equation might now be represented by only ($n - 1$) distinct characters; and therefore, of the subordinate equations derived from the construction of the coefficients, *two* might be employed to determine *one root.* (a and b) being equal, the equation furnished by the value of the coefficient (p), and also that furnished by the coefficient (q), may be *both together* used to determine the same quantity. But, if any quantity (a) be a root of an equation, the simple equation ($x - a = 0$) must be a divisor of that equation;† therefore *here* ($x - a$) must be a common divisor of the two equations furnished by (p and q),

* Vide MACLAURIN's Algebra, chap. iv. p. 162, *et infra.*

† Vide SANDERSON's Algebra, Vol. ii. p. 679, 680, Art. 432, and all algebras on the method of divisors.

and consequently may be found, without *resolving* either of them, by continual division or subtraction, according to the ordinary rule for finding the common measure.*

12. Any other relation *from the knowledge of which one character may be made to represent two or more roots*, evidently answers the same end. Indeed all relations of that kind may be converted into equality itself, by taking, instead of the given equation, some other properly derived from it. Thus if, instead of (a and b) being the same, (b) had been supposed the negative of (a) or ($-a$), and then, instead of the former equation, that of the squares of the roots were taken, the relation would be made equality; for (a) and ($-a$) have the same square. If arithmetical proportion was known to be the relation of any number of the roots, by taking the equation of their differences, it would also be converted into equality.

13. If three or more roots, or any number of parcels of roots, are known to be related, and their common relation be used to represent them, of course the number of distinct characters to be determined will proportionably be diminished: and, as the number of subordinate equations furnished by the coefficients remains always the same, while the dimension of the proposed equation is unaltered, more of them may be used together to discover the related roots, and their investigation be proportionably facilitated. *This single observation*, in the hands of a

* Vide SANDERSON's Algebra, (quarto ed. Vol. i. p. 86, 87, 88,) where the rule is well given; and MACLAURIN's Algebra, (P. II. cap. iv. p. 162.); or Mr. HELLINS's Essay upon the Reduction of Equations having equal Roots. But of the last it should be observed, that some qualification must be made to the assertion, that the reduction may be carried on till a *simple* equation is obtained. In cases where there is only *one* pair of roots equal, that proposition is undoubtedly true; but,

skilful analyst, is sufficient for the reduction, if not the solution, of any particular *numeral* equation whatsoever, and the more so the larger its dimension: for, from the endless variety of relations numbers bear to each other, hardly any set of them can occur, as the coefficients of an equation, or perhaps exist, that, upon being compared, do not exhibit some peculiarity (of greater or less extent) sufficient to afford a clue to the correspondent relation in their roots. And, if no such clue is immediately given by the equation itself, taking the equation of the differences or sums in pairs, or of the squares, *&c.* of the roots, will soon find one. But, as peculiarities of that sort (though never so frequent) may be deemed always accidental, and evidently, no general method can be founded upon them, *even where the coefficients are given*, it may be asked, How any use can be made of them in cases of indeterminate equations?

14. To this I answer, that there are some properties of quantities that depend only on the index of the equation, without any regard to the value of its coefficients; or, in other words, there are some peculiar properties which merely depend upon the *number* of any set of quantities, abstracted from all consideration of their nature and values. For example, two quantities (a) and (b) have their differences the same quantity ($a - b$), only taken both affirmatively and negatively, ($a - b$) and ($b - a$); when squared, these differences become equal; ($a^2 - 2ab + b^2$) is the square of both: therefore, let the quantities themselves be chosen as they may, the equation of the squares of their

if 2, 3, or more pairs of roots are equal, the reduction can only be carried down to a quadratic, cubic, *&c.* for, *every* pair of equal roots being equally to be found by the method, of course the final or resulting equation must be of a dimension as great as their number.

differences *must* have *both equal roots*, and consequently be reducible by the reasoning in Art. 11, 12, and 13. Again, *three* quantities, however distinct in themselves, give a set of differences marked with a peculiar relation, any two of them being equal to the third; (a, b, c,) being three quantities, ($\overline{a-b}+\overline{b-c}=\overline{a-c}$). Also, if the three quantities be so chosen originally as to have their sum equal nothing, one of them must necessarily equal in magnitude the sum of the remaining two; and therefore, whether taken simply or summed in pairs, their relative magnitudes must remain the same. Again, *four* quantities, of any sort whatever, may be pursued to a constant relation, though somewhat more remote, and grounded upon very different causes; *viz.* (a, b, c, d,) being four quantities, from their combinations by pairs (ab, ac, ad, bc, bd, cd,) six in number, added together two by two, thus,

$$(ab+cd)$$
$$(ac+bd)$$
$$(ad+bc)$$

three quantities are formed, sufficiently distinguished from the group of similar combinations to be found separately, as will be shewn hereafter. And also, if the four quantities are originally so taken as to have their sum equal to nothing, their sums in pairs, though six in number, will be reduced to three in effect; for, if ($a+b+c+d=0$), by transposition,

$$(a+b=-c-d)$$
$$(a+c=-b-d)$$
$$(a+d=-b-c)$$

i. e. three of the six must be merely the negatives of the other three; which relation, if they are squared, will become equality, so that the number of distinct squares will be only three. These

properties, though without any order or connection, and confined merely to particular ranks or numbers of quantities, being general to *all possible* or *imaginable* quantities of *those classes*, afford methods general, as to those degrees, but without producing any result *really general to equations at large.*

15. Having shewn that an indeterminate general equation cannot be resolved by any of the methods whose principle is yet known, because they are all grounded upon the assumption of some particularity, either inherent in the roots, or universally communicable to them, which, so far from being general, is seldom found, and absolutely incompatible with many sorts of roots; that the difficulty is in all cases the same,—the intrusion of superfluous roots and higher radicals; that a relation of any kind (when known) obviates that difficulty, as far as it extends; and that *some* orders of quantities have generally a *constant* and *necessary* relation, more or less remote, I proceed to examine, more minutely, the application of these observations to the several degrees of equations to which they materially apply.

CHAP. II.

Of the Resolution or Reduction of Equations of particular Degrees.

16. In examining those degrees of equations which submit to be resolved, I shall observe the same order as I did before; *i. e.* first consider the power of obtaining a general formula, or complete resolution; and, if that is not attainable directly, inquire by what general means the roots can be separately inves-

tigated, and what new forms they have taken, or what different functions of them are used in the operation.

17. If we resume the general indeterminate equation ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3}$ &c. $= 0$), and assign the progressive values (2, 3, 4 &c.) to the index (n), in the first case it will become the quadratic ($x^2 - px + q = 0$). Now, as this equation has two roots, in order to obtain a general formula for its resolution, the first step that suggests itself is, to inquire what is necessary to construct a general representation of two quantities in a simple equation. Two quantities are known to be generally expressed by means of their sum and difference; that half their sum added to half their difference gives the greater, and the same quantities subtracted, the lesser. The sum being always the coefficient of the second term of the equation, *is given in all cases*, and *here* the difference is readily found; for, the square of the difference of any two quantities differs from the square of their sum by a constant quantity, *viz.* four times their product or the coefficient of the third term. If (a) and (b) be called the roots of the equation ($x^2 - px + q = 0$); then ($p = a + b$) and ($p^2 = a^2 + 2ab + b^2$),
($q = ab$) and ($-4q = \quad -4ab \quad$),

$$\text{whence } \overline{(a^2 - 2ab + b^2} = \overline{a - b}|^2 = \overline{p^2 - 4q} \left\{ \begin{array}{l} \text{the square} \\ \text{of the dif-} \\ \text{ference.} \end{array} \right.$$

The difference itself is therefore ($\sqrt{p^2 - 4q}$). And now, being possessed of the parts required to construct a general representation of the two quantities, we can at once complete the formula of general resolution of equations of this degree, *viz.*

$$\left(x = \frac{p \pm \sqrt{p^2 - 4q}}{2}\right).$$

This, as I observed before in Art. 6, is however the *same* qua-

dratic, only reverted; for, the quadratic surd it contains is frequently incapable of further reduction. Therefore, generally speaking, the *degree* of the equation is not altered; only the place of the index, which being first affixed to the unknown quantity, is now transferred to the known ones. But nevertheless, this resolution is, in *all* cases, *equally true* and *direct;* for, involving no other radical than belongs to the degree it relates to, it faithfully exhibits the nature of the roots, and is always rational or real, or not, according as they are so.

18. If, instead of seeking, *a priori*, the formula of resolution, we attempt to find the roots simply, we may instantly trace a constant connection between them, or at least between their differences; which (however the quantities are varied) are always related in the same manner, being $(a - b)$ and $(-a + b)$ the same quantity with different signs, and consequently their squares precisely the same. From which it appears, that the equation of those differences will always want the second term or be a pure quadratic; and *that* of their squares will be a perfect binomial square, having both roots equal; which roots may therefore, by the reasoning in Art. 11, be certainly found. But the inference is just the same as before: the equation is not lowered in degree; the equal relation is brought no nearer than between the squares of the differences; and, when *they* are found, the same quadratic surd must be used to arrive at the roots themselves. This formula of resolution $\left(x = \frac{p \pm \sqrt{p^2 - 4q}}{2}\right)$ is the same given for quadratics in every algebra; but it is not usually remarked, or perhaps understood, that the whole operation, however varied in appearance by setting about to complete the square (as it is called) or to destroy the second term,

is merely employed to obtain the difference of the roots; that (upon analysing the formula) the part under the vinculum is always that difference, and nothing else, and *why* it must be so.

19. Next, let $(n = 3)$, and the equation be the complete cubic $(x^3 - px^2 + qx - r = 0)$. If we make it our first step here, as in the last case, to inquire what is necessary to construct a general representation of three numbers in simple equations, we shall find it must consist of the same parts, the sum and the differences: but, as the differences increase in number, to show the order in which they are taken, and the law they observe progressively, I shall subjoin a general table of the simple representation of the different orders of quantities. As in every equation the sum of the roots is always given, I shall, for greater simplicity in my table, suppose it always to vanish. If then there be a series of general equations, beginning with a quadratic, and proceeding upwards with progressive indexes, in all of which the coefficient of the second term (p) be taken $= (0)$, and (A) be supposed a difference of the roots of the first, (A) and (B) two of the differences of the roots of the second, $(A, B, C,)$ three differences of those of the third, and so on; in taking of which differences, no other caution is necessary than that they should be similarly situated, *viz.* all derived by comparing the same individual root with the remaining ones, as if (a) be taken as a root, and $(\overline{a - b})$ be the first difference, $(\overline{a - c}, \overline{a - d}, \overline{a - e}$ &c. . . . $\overline{a - n})$, having *all* the *same antecedent* letter (whose number will always be $(n - 1)$), must be the rest; *then*, the table will be as follows:

Table of the simple Representation of the Roots of Equations of progressive Indexes.

In quadratics,

$$x = \pm \frac{A}{2}$$

In cubics,

$$x = \begin{cases} \dfrac{A + B}{3} \\[2ex] \dfrac{-A - B}{2 \cdot 3} + \begin{cases} + \dfrac{A - B}{2} \\[2ex] - \dfrac{A + B}{2} \end{cases} \end{cases}$$

In biquadratics,

$$x = \begin{cases} \dfrac{A + B + C}{4} \\[2ex] - \dfrac{A - B - C}{3 \cdot 4} \begin{cases} + \dfrac{A - B}{3} - \dfrac{B + C}{3} \\[2ex] + \dfrac{A - C}{3} + \dfrac{B - C}{3} \\[2ex] - \dfrac{A + B}{3} - \dfrac{A + C}{3} \end{cases} \end{cases}$$

In the fifth degree,

$$x = \begin{cases} \dfrac{A + B + C + D}{5} \\[2ex] - \dfrac{A - B - C - D}{4 \cdot 5} \begin{cases} + \dfrac{A - B}{4} - \dfrac{B + C}{4} - \dfrac{B + D}{4} \\[2ex] + \dfrac{A - C}{4} + \dfrac{B - C}{4} - \dfrac{C + D}{4} \\[2ex] + \dfrac{A - D}{4} + \dfrac{B - D}{4} + \dfrac{C - D}{4} \\[2ex] - \dfrac{A + B}{4} - \dfrac{A + C}{4} - \dfrac{A + D}{4} \end{cases} \end{cases}$$

In the (nth) degree, or generally,

$$x = \begin{cases} \dfrac{A + B + C + D + E \text{ \&c.} \ldots \overline{(n-1)}}{n} \text{ in number} \\[2ex] \dfrac{-A - B - C - D - E \text{ \&c.}}{\overline{n-1} \times n} \begin{cases} + \dfrac{A-B}{n-1} - \dfrac{B+C}{n-1} \text{ \&c. } \overline{n-2} \text{ in number} \\ + \dfrac{A-C}{n-1} + \dfrac{B-C}{n-1} \text{ \&c.} \\ + \dfrac{A-D}{n-1} + \dfrac{B-D}{n-1} \text{ \&c.} \\ + \dfrac{A-E}{n-1} + \dfrac{B-E}{n-1} \text{ \&c.} \end{cases} \end{cases}$$

---

20. The inspection of the table shews us, that in *all* cases, to construct a *general simple representation* of any number of quantities, and consequently to construct a *direct resolution* of their equation, we must first find a certain number of their differences; but we have no *general means* of separating particular differences from the rest; and the whole number of differences increases in a proportion so much greater than the number of quantities, that the former difficulty recurs, the previous steps involve higher dimensions than the original equation. The original index being (n), that of the equation of the difference of the roots is ($n \times \overline{n-1}$). However, from the nature of differences, (being taken both affirmatively and negatively,) all equations formed from them must (as observed of quantities of that sort in Art. 11.) be universally *deficient* in *every alternate* term, which brings their equation to the *form* of equations of *only half* their own index, or $\left(n \times \dfrac{\overline{n-1}}{2}\right)$: but, in this case, their differences are six, and their equation, *with that consideration*, is reduced no lower than a cubic form, which is the same degree with the proposed equation; therefore, it does not appear

that we can be enabled, *a priori*, to determine the formula of any direct resolution of this case.

21. Let us then try to trace some relation which may convert some or all of the roots, or some regular function of them, into equal quantities; when, the equation of that function having equal roots, of course those roots will be separately deducible, as shewn in Art. 11, 12. In Art. 14, we may remember, two particularities were mentioned to belong to three quantities, *viz.* that their differences were so related as to be every two of them equal to the third; and that, if the quantities themselves have their sum equal to nothing, two of *them* also must equal the third, and their magnitude be respectively the same, whether they are taken simply, or summed in pairs. To avail ourselves of *both* these properties, let us suppose the second term to be expunged from the given equation, (which we know may always be effected,) its form will then be ($x^3 - qx + r = 0$),* and the sum of its roots equal to nothing. Let (a) and (b) be two of its roots, the third will therefore be ($-a-b$); take their sums by two ($-a$, $-b$, $a+b$); take their differences ($2a+b$, $a+2b$, $a-b$) *and their negatives*, which may be divided into two sets whose sum is nothing, like that of the roots, *viz.* $\left\{\begin{array}{ccc} a-b, & a+2b, & -2a-b \\ -a+b, & -a-2b, & 2a+b \end{array}\right\}$.

So that, from the given equation we derive three others, which make a set of four exactly similar.

* Besides expunging (p), the sign of (q) has been changed; because, in cases of real roots, it will invariably become negative upon destroying the second term. Vide note in p. 275.

1st. $(x^3 - qx + r = 0)$ the given equation.

2d. $(x^3 - qx - r = 0)$ that of its roots summed in pairs.

$$\left.\begin{array}{ll}\text{3d.} & x^3 - 3qx + \sqrt{4q^3 - 27r^2} = 0 \\ \text{4th.} & x^3 - 3qx - \sqrt{4q^3 - 27r^2} = 0\end{array}\right\}\text{two similar equations,}$$

formed by dividing $(x^6 - 6qx^4 + 9q^2x^2 - \overline{4q^3 + 27r^2} = 0)$, the equation of the differences, into two wanting the second term.

22. Now, leaving these considerations for a moment, let us speculate upon the further reduction of the equation. If, instead of the present form $(x^3 - qx + r = 0)$, (q) could be supposed to vanish as well as (p), a still more powerful additional relation would be given the roots; for, the equation being *then* a pure cubic, $(x^3 = \pm r)$, its roots would obviously be the cube roots of (r), and *all cube roots* are *alike*. If (r) be a cube, and $(\sqrt[3]{r})$ be one of its roots, the remaining two are $\left(\frac{-1+\sqrt{-3}}{2} \times \sqrt[3]{r} \text{ and } \frac{-1-\sqrt{-3}}{2} \times \sqrt[3]{r}\right)$, let (r) be any quantity whatsoever, real or imaginary. But it is clear, from what has been before observed in Art. 9, that this reduction is not generally possible, since it supposes two contiguous intermediate terms to vanish together, which real roots do not admit of: it must therefore be effected by means of some imaginary assumption. Those who are conversant in the use of impossible quantities, will at once perceive, that the addition or subtraction (which in surd quantities is always the same thing, as they are equivocal in sign,) of the imaginary surd $\left(\sqrt{-\frac{q}{3}}\right)$ to each root of the equation, will infallibly cause (q) to vanish, but the new roots $\left\{\begin{array}{l} a + \sqrt{-\frac{1}{3}q}, \\ b + \sqrt{-\frac{1}{3}q}, \end{array}\right. - a - b + \sqrt{-\frac{1}{3}q})$ so formed, would not have their sum equal to nothing; and

therefore, in destroying the third term, the second would be revived, so that nothing would be gained.

23. To understand how this difficulty is ever removed, let us examine particularly some equation that wants both second and third terms, and observe accurately the constitution of its roots. The simplest of the kind is the pure cubic ($x^3 = 1$), whose roots are $\left(1, \frac{-1 \pm \sqrt{-3}}{2}\right)$; but, to avoid fractions in the roots, let us take ($x^3 = 8$), whose roots are ($2, -1 \pm \sqrt{-3}$). Distinguishing the real and the imaginary parts, the real are ($2, -1, -1$); the imaginary are ($\pm \sqrt{-3}$ or $\pm 3 \times \sqrt{-\frac{1}{3}}$), which are the *differences of the real parts*, multiplied by the imaginary surd ($\sqrt{-\frac{1}{3}}$). It appears, therefore, that the roots of a pure cubic are compounded of the roots of some affected cubic, added to their differences drawn into the imaginary surd ($\sqrt{-\frac{1}{3}}$). The real parts ($2, -1, -1$) are the roots of the cubic equation ($x^3 - 3x + 2 = 0$). The imaginary, of the equation ($x^3 + 3x + \ast = 0$), or the roots of the *similar* equation of the differences of the former, *viz.* ($x^3 - 9x + \ast = 0$), drawn into the ($\sqrt{-\frac{1}{3}}$); and, from *their* addition are formed the roots of the pure cubic ($x^3 = 8$). In constructing which, it is material to observe, *each root* of the first equation is joined to the difference of the remaining pair; but it may be remembered, that *three* quantities whose sum is nothing, are the same when summed in pairs, *i. e.* each is (in quantity) the sum of the other two, therefore, each difference is in fact added to the *sum of the same quantities;* and, if the question were proposed to reduce the equation ($x^5 - 3x + 2 = 0$) to a pure cubic, the rule furnished by this example would be, to find the equation of the sum of its roots in pairs, which, by the last Article, is

$(x^3 - 3x - 2 = 0)$; to find the *similar* equation of their differences $(x^3 - 9x + * = 0)$; and, to find the equation produced by the quantities formed from the addition of the roots of the one to those of the other multiplied into the imaginary surd $(\sqrt{-\frac{1}{3}})$. The equation last found would, however, be of the dimension of the 9th power, at least: for, the addition of each root of the second equation to every separate root of the first, produces a separate quantity: thus,

$$\left.\begin{array}{l} (2, -1, \quad - \quad 1) \\ \textit{being the roots of the 1st.} \\ (0, +\sqrt{-3}, -\sqrt{-3}) \\ \textit{those of the 2d.} \quad - \end{array}\right\} \left\{\begin{array}{lll} \overline{2+0} & ,\overline{-1+0} & ,\overline{-1+0} \\ 2+3\sqrt{-\tfrac{1}{3}}, & -1+3\sqrt{-\tfrac{1}{3}}, & -1+3\sqrt{-\tfrac{1}{3}} \\ 2-3\sqrt{-\tfrac{1}{3}}, & -1-3\sqrt{-\tfrac{1}{3}}, & -1-3\sqrt{-\tfrac{1}{3}} \end{array}\right\}$$

will be the nine quantities formed by their addition. But we have a decisive clue to distinguish *some* from the rest; for we know, that if we find the equation of the cubes of those quantities, *it must have three equal roots*; for, every time the *sum* of two of the roots of the first equation meets its *own* difference, it will constitute a cube root of (8), and therefore, the equation $(x^3 - 8 = 0)$ will be three times contained in the resulting equation of cubes. That equal root being discovered by the method of finding equal roots, so often alluded to before, reduces the equation $(x^3 - 3x + 2 = 0)$ to the pure cubic $(x^3 = 8)$.

24. The instance in the last article, of the reduction of the equation $(x^3 - 3x + 2 = 0)$ to a pure cubic, by means of the equation $(x^3 + 3x = 0)$, evidently depends upon the coefficient of the second term vanishing; and also, that of the third term being the same in both, but of opposite signs. For, the roots of the one, in their combinations by two, producing (-3), and those of the other $(+3)$, of course destroy each

other; and, as the sums of both equal nothing, when added together their sum will still be nothing; so that *no new second term can arise*, as in Art. 22. If we now return to the considerations in Art. 21, where we shewed how to derive from every cubic equation ($x^3 - qx + r = 0$) wanting the 2d term, a similar equation ($x^3 - 3qx + \sqrt{4q^3 - 27r^2} = 0$), being the equation of three of the differences of the roots of the former, so arranged as to want the second term also, we may perceive that, to render the third term the same in both, we need only divide the roots of the latter by ($\sqrt{3}$), or, which is the same thing, multiply them into the ($\sqrt{\frac{1}{3}}$). For, the equation ($x^3 - 3qx + \sqrt{4q^3 - 27r^2} = 0$), when its roots are multiplied by the ($\sqrt{\frac{1}{3}}$), becomes $\left(x^3 - qx + \frac{\sqrt{4q^3 - 27r^2}}{3\sqrt{3}} = 0\right)$.* If, by the same reason, they had been multiplied by the ($\sqrt{-\frac{1}{3}}$), it would be $\left(x^3 + qx + \frac{\sqrt{4q^3 - 27r^2}}{3\sqrt{-3}} = 0\right)$; where the sign of the coefficient of (x) is opposite to that of (q) in the given equation. Therefore, the roots of the equation ($x^3 - qx + r = 0$), and that of its differences, multiplied into the imaginary surd $\sqrt{-\frac{1}{3}}$, *viz.* $\left(x^3 + qx + \frac{\sqrt{4q^3 - 27r^2}}{3\sqrt{-3}} = 0\right)$, will, by being added together, (according to the method in the last Article,) lead to a reduction of that equation to a pure cubic; *i. e.* the equation formed from their addition will have three roots, whose cubes are the same.

25. The analysis of the pure cubic gives us the following *general properties*, belonging to any set of those equations whose sum is nothing.

Viz. 1st. If three such quantities (a, b, $-a-b$) be added

* Vide SANDERSON's Algebra, Vol. II. p. 688; and HALE's Analysis, p. 146.

in pairs, and three of their differences be also taken so as to have their sum nothing, $(a - b,\ a + 2b,\ -2a - b)$; if *then* each sum be formed into a binomial, by joining to it its correspondent difference multiplied by the imaginary surd $\sqrt{-\frac{1}{3}}$, the quantities so formed

$$a + b + \overline{a - b}|\sqrt{-\tfrac{1}{3}}$$
$$-a + \overline{a + 2b}|\sqrt{-\tfrac{1}{3}}$$
$$-b - \overline{2a - b}|\sqrt{-\tfrac{1}{3}}$$ will have the *same cube*.

Example 1st,

$$\overline{a+b} + \overline{a-b}|\sqrt{-\tfrac{1}{3}}$$
$$\overline{a+b} + \overline{a-b}|\sqrt{-\tfrac{1}{3}}$$
$$\overline{a+b}|^2 + 2\overline{a+b}|\times\overline{a-b}|\sqrt{-\tfrac{1}{3}}$$
$$-\frac{\overline{a-b}|^2}{3}$$

$$\frac{2a^3+8ab+2b^2}{3} + 2.\overline{a+b}\times\overline{a-b}|\sqrt{-\tfrac{1}{3}} = \ldots \overline{a+b+\overline{a-b}|\sqrt{-\tfrac{1}{3}}}|^2$$
$$\overline{a+b} + \overline{a-b}\sqrt{-\tfrac{1}{3}}$$

$$\frac{2a^3+10a^2b+10ab^2+2b^3}{3} + 2.\overline{a+b}|^2\times\overline{a-b}|\sqrt{-\tfrac{1}{3}}$$
$$+\frac{2.\overline{a+b}|^2\times\overline{a-b}|}{3}\sqrt{-\tfrac{1}{3}}$$
$$-\frac{2a^3+2a^2b+2ab^2-2b^3}{3} \qquad + \frac{4ab}{3}\times\overline{a-b}|\sqrt{-\tfrac{1}{3}}$$

☞ $$*\frac{12a^2b+12ab^2}{3}* + \frac{8a^3+12a^2b-12ab^2-8b^3}{3\sqrt{-3}}\Bigg| = \overline{\overline{a+b}|+\overline{a-b}|\sqrt{-\tfrac{1}{3}}}|^3$$

$$-a + \overline{a+2b}|\sqrt{-\tfrac{1}{3}}$$
$$-a + \overline{a+2b}|\sqrt{-\tfrac{1}{3}}$$
$$a^2 - \overline{2a^2-4ab}|\sqrt{-\tfrac{1}{3}}$$
$$\frac{-a^2-4ab-4b^2}{3}$$

$$\frac{2a^2-4ab-4b^2}{3} - \overline{2a^2-4ab}|\sqrt{-\tfrac{1}{3}} = \ldots \overline{-a+\overline{a+2b}|\sqrt{-\tfrac{1}{3}}}|^2$$

$$\frac{2a^2-4ab-4b^2}{3} - \overline{2a^2-4ab}\,\sqrt{-\tfrac{1}{3}}$$

$$-a+\overline{a+2b}\,\sqrt{-\tfrac{1}{3}}$$

$$-\frac{2a^3+4a^2b+4ab^2}{3} + \overline{2a^3+4a^2b}\,\sqrt{-\tfrac{1}{3}}$$

$$\frac{2a^3+8a^2b+8ab^2}{3} - \frac{\overline{a+2b}^3}{3}\,\sqrt{-\tfrac{1}{3}}$$

$$+a^2\times\overline{a+2b}\,\sqrt{-\tfrac{1}{3}}$$

☞ $$* \; \frac{12a^2b+12ab^2}{3}+\frac{8a^3+12a^2b-12ab^2-8b^3}{3\sqrt{-3}} = \ldots \overline{-a+\overline{a+2b}\,\sqrt{-\tfrac{1}{3}}}^{\,3}$$

$$-b-\overline{2a-b}\,\sqrt{-\tfrac{1}{3}}$$

$$-b-\overline{2a-b}\,\sqrt{-\tfrac{1}{3}}$$

$$b^2+\overline{4ab+2b^2}\,\sqrt{-\tfrac{1}{3}}$$

$$-\frac{b^2-4ab-4a^2}{3}$$

$$\frac{2b^2-4ab-4a^2}{3}+\overline{4ab+2b^2}\,\sqrt{-\tfrac{1}{3}} = \ldots\ldots \overline{-b-\overline{2a-b}\,\sqrt{-\tfrac{1}{3}}}^{\,2}$$

$$-b-\overline{2a-b}\,\sqrt{-\tfrac{1}{3}}$$

$$-\frac{2b^3+4ab^2+4a^2b}{3} - \overline{4ab^2-2b^3}\,\sqrt{-\tfrac{1}{3}}$$

$$\frac{2b^3+8ab^2+8a^2b}{3}+\frac{\overline{2a+b}^3}{3}\,\sqrt{-\tfrac{1}{3}}$$

$$+b^2\times -\overline{2a-b}\,\sqrt{-\tfrac{1}{3}}$$

☞ $$* \; \frac{12a^2b+12ab^2}{3}+\frac{8a^3+12a^2b-12ab^2-8b^3}{3\sqrt{-3}} = \ldots \overline{-b-\overline{2a-b}\,\sqrt{-\tfrac{1}{3}}}^{\,3}$$

The quantity

$$\frac{12a^2b+12ab^2}{3}+\frac{8a^3+12a^2b-12ab^2-8b^3}{3\sqrt{-3}} = 4\times\overline{a^2b+ab^2 + \frac{2a^3+3a^2b-3ab^2-2b^3}{3\sqrt{-3}}}$$

is therefore the common cube of those 3 binomials. *Q. E. M.*

Now let the equation of the 3 quantities $(a,\ b,\ -a-b,)$ be $(x^3-qx+r=0)$; then, by the construction of equations,

$(q=a^2+ab+b^2)$ and $(4q^3=4a^6+12a^5b+24a^4b^2+28a^3b^3+24a^2b^4+12ab^5+4b^6)$, also $(-r=a^2b+ab^2)$ and $(27r^2=\text{- - - - - - - -}\ 27a^4b^2+54a^3b^3+27a^2b^4)$, whence $(4q^3-27r^2=4a^6+12a^5b-3a^4b^2-26a^3b^3-3a^2b^4+12ab^5+4b^6)$ and its square root, or $(\sqrt{4q^3-27r^2}=\overline{2a^3+3a^2b-3ab^2-2b^3})$. Therefore, the quantity

$$4\times\overline{a^2b+ab^2}+\frac{2a^3+3a^2b-3ab^2-2b^3}{3\sqrt{-3}}=4\times\overline{-r+\frac{\sqrt{4q^3-27r^2}}{3\sqrt{-3}}}.$$

The equation $(x^3-qx+r=0)$ may hence be reduced to a pure cubic of this form $\left(x^3=4\times\overline{-r+\frac{\sqrt{4q^3-27r^2}}{3\sqrt{-3}}}\right)$, which, when cleared of its irrational quadratic surd, becomes $\left(x^6+8rx^3+16r^2=\frac{-64q^3+432r^2}{27}\right)$, or $\left(x^6+8rx^3+\frac{64q^3}{27}=0\right)$; or, dividing its roots by (2) to reduce it still lower, $\left(x^6+rx^3+\frac{q^3}{27}=0\right)$, the common reducing equation obtained by CARDAN's rule.

Example 2d. Let $(1,2,-3)$, the roots of the equation $(x^3-7x+6=0)$, be taken; the binomials formed from them will be, according to the directions of the rule,

$$\left.\begin{array}{l}1+2=+3\\1-2=-1\end{array}\right\}\ \underline{3-1\sqrt{-\tfrac{1}{3}}}$$

$$\left.\begin{array}{l}1-3=-2\\1+3=+4\end{array}\right\}\ \underline{-2-4\sqrt{-\tfrac{1}{3}}}$$

N. B. It is necessary to change the sign of the middle difference, because their sum must always $=0$).

$$\left.\begin{array}{l}2-3=-1\\2+3=+5\end{array}\right\}\ \underline{-1+5\sqrt{-\tfrac{1}{3}}}$$

$$\left\{\begin{array}{ccc}
3-1\sqrt{-\tfrac{1}{3}} & -2-4\sqrt{-\tfrac{1}{3}} & -1+5\sqrt{-\tfrac{1}{3}}\\
3-1\sqrt{-\tfrac{1}{3}} & -2-4\sqrt{-\tfrac{1}{3}} & -1+5\sqrt{-\tfrac{1}{3}}\\
\hline
9-6\sqrt{-\tfrac{1}{3}} & 4+16\sqrt{-\tfrac{1}{3}} & 1-10\sqrt{-\tfrac{1}{3}}\\
-\tfrac{1}{3} & -\tfrac{16}{3} & -\tfrac{25}{3}\\
\hline
\tfrac{26}{3}-6\sqrt{-\tfrac{1}{3}} & -\tfrac{4}{3}+16\sqrt{-\tfrac{1}{3}} & -\tfrac{22}{3}-10\sqrt{-\tfrac{1}{3}}\\
3-1\sqrt{-\tfrac{1}{3}} & -2-4\sqrt{-\tfrac{1}{3}} & -1+5\sqrt{-\tfrac{1}{3}}\\
\hline
26-18\sqrt{-\tfrac{1}{3}} & \tfrac{8}{3}-32\sqrt{-\tfrac{1}{3}} & \tfrac{22}{3}+10\sqrt{-\tfrac{1}{3}}\\
-2-\tfrac{26}{3}\sqrt{-\tfrac{1}{3}} & \tfrac{64}{3}+\tfrac{16}{3}\sqrt{-\tfrac{1}{3}} & \tfrac{50}{3}-\tfrac{110}{3}\sqrt{-\tfrac{1}{3}}\\
\hline
24-\tfrac{80}{3}\sqrt{-\tfrac{1}{3}} & 24-\tfrac{80}{3}\sqrt{-\tfrac{1}{3}} & 24-\tfrac{80}{3}\sqrt{-\tfrac{1}{3}}
\end{array}\right.$$

The quantity $\left(24-\frac{80}{3\sqrt{-3}}\right)$ is therefore the common cube of these 3 binomials.

Example 3d Let $(-1, -4, +5)$, the roots of the equation $(x^3 - 21\,x - 20 = 0)$, which are the differences used in the last example, be next taken;

$$\left.\begin{array}{l} -1-4=-5 \\ -1+4=+3 \end{array}\right\} -5+3\sqrt{-\tfrac{1}{3}} = -5+1\sqrt{-3}$$

$$\left.\begin{array}{l} -1+5=+4 \\ -1-5=-6 \end{array}\right\} +4+6\sqrt{-\tfrac{1}{3}} = \ 4+2\sqrt{-3}$$

$$\left.\begin{array}{l} -4+5=+1 \\ -4-5=-9 \end{array}\right\} +1-9\sqrt{-\tfrac{1}{3}} = \ 1-3\sqrt{-3}$$

$-5+1\sqrt{-3}$	$4+2\sqrt{-3}$	$1-3\sqrt{-3}$
$-5+1\sqrt{-3}$	$4+2\sqrt{-3}$	$1-3\sqrt{-3}$
$25-10\sqrt{-3}$	$16+16\sqrt{-3}$	$1-6\sqrt{-3}$
-3	-12	-27
$22-10\sqrt{-3}$	$4+16\sqrt{-3}$	$-26-6\sqrt{-3}$
$-5+1\sqrt{-3}$	$4+2\sqrt{-3}$	$1-3\sqrt{-3}$
$-110+50\sqrt{-3}$	$16+64\sqrt{-3}$	$-26-6\sqrt{-3}$
$30+22\sqrt{-3}$	$-96+8\sqrt{-3}$	$-54+78\sqrt{-3}$
$-80+72\sqrt{-3}$	$-80+72\sqrt{-3}$	$-80+72\sqrt{-3}$

which last cube, if divided by (3), becomes $(-\frac{80}{3} + 24\sqrt{-3})$, or exactly the reverse of the first; the reason of which will be shewn in the next section of this Article.

The cube $(24 - \frac{80}{3}\sqrt{-\frac{1}{3}})$, when its equation $(x^3 = 24 - \frac{80}{3}\sqrt{-\frac{1}{3}})$ is made rational, gives the quadratic-formed equation of the 6th degree $(x^6 - 48\,x^3 - + 576 = -\frac{6400}{27})$; or, transposing all the terms to one side, and dividing it by (2), to reduce it, as before, $(x^6 - 6x^3 + \frac{343}{27} = 0)$; the same equation that results from the common methods.

2dly. The differences of the three differences $(a - b,\ a + 2b,\ -2a - b)$ are $(3\,a,\ 3\,b,\ 3 \times \overline{a + b})$, or merely three times the original quantities. Had, therefore, the differences themselves been taken as original quantities, and binomials been formed from *them*, according to the directions before observed, *those binomials*, and the ultimately resulting cubes, would differ from the former, in nothing essential but the place of the surd. The differences which were affected with it before, would *now* be

clear; and the quantities themselves, or, which is the same thing, their sums in pairs, be affected with it. However, as these latter are to be multiplied by three, *that multiplication* will destroy the fraction when they come again to be multiplied by the surd $(\sqrt{-\frac{1}{3}})$, since $(3 \times \sqrt{-\frac{1}{3}} = \sqrt{-3})$. Wherefore, the same end, as to reducing the equation, will be obtained, *whether, after adding the sums* of the roots in pairs to their *respective differences,* we *multiply the sums by* $(\sqrt{-3})$, or *divide* the *differences* by it; as has been already shewn in the 3d Example to the last Section of this Article.

3dly. If any cubic equation wanting the second term, be transformed into the equation of that function of its roots, formed of the cubes of the binomials arising from joining the sum of each pair of roots to its correspondent difference drawn into the imaginary fractional surd $(\sqrt{-\frac{1}{3}})$, or each difference to its correspondent sum drawn into the surd $(\sqrt{-3})$, the transformed equation will have among its roots three equal cubes; by finding which, according to the methods of finding equal roots, the equation is reduced to a pure cubic.

4thly. The roots of a cubic equation may be *all* real; or only one of them real, and the remaining *two* imaginary. If only one be real, they will be of this form $\left(a, -\frac{a \pm b\sqrt{-1}}{2}\right)$; and, by taking their sums and differences according to the rule, and multiplying the latter into the $(\sqrt{-\frac{1}{3}})$, *one* of the resulting binomials will be real, and the other two imaginary: the cube produced by them will therefore be real. When all the roots are real, *if two be equal, one* difference necessarily vanishes; wherefore, the imaginary factor will only appear about the two

that remain; and here again, the cube produced will be real. But, if all the roots are real, and unequal, their sums and differences will *all* be real: whence, *all* the binomials will involve the imaginary surd; which constitutes the irreducible case.

To give examples of this, let, 1st. $(x^3 - 2x + 4 = 0)$, a cubic equation, whose roots are $(2, -1 + \sqrt{-1}, -1 - \sqrt{-1})$; the binomials constructed by taking their sums and differences as before, will be

$$\left.\begin{aligned} 2 - \overline{1 + \sqrt{-1}} &= \overline{1 + \sqrt{-1}} \\ 2 + \overline{1 - \sqrt{-1}} &= \overline{3 - \sqrt{-1}} \end{aligned}\right\}$$

$$\left.\begin{aligned} 2 - \overline{1 - \sqrt{-1}} &= \overline{1 - \sqrt{-1}} \\ 2 + \overline{1 + \sqrt{-1}} &= \overline{3 + \sqrt{-1}} \end{aligned}\right\}$$

$$\left.\begin{aligned} \overline{-1 + \sqrt{-1}}\;\overline{-1 - \sqrt{-1}} &= -2 \\ -1 + \sqrt{-1} + 1 + \sqrt{-1} &= 2\sqrt{-1} \end{aligned}\right\}$$, which last binomial $\overline{-2 + 2\sqrt{-1}}$, when the latter quantity $(2\sqrt{-1})$ is drawn into the imaginary surd $(\sqrt{-\tfrac{1}{3}})$, becomes $\left(-2 - \frac{2}{\sqrt{3}}\right)$, a real quantity.

2dly. Let $(x^3 - 3x + 2 = 0)$ be proposed, whose roots have been, in Art. 23, shewn to be $(2, -1, -1)$. Here

$$\left.\begin{aligned} 2 - 1 &= +1 \\ 2 + 1 &= +3 \end{aligned}\right\}$$

$$\left.\begin{aligned} 2 - 1 &= +1 \\ 2 + 1 &= +3 \end{aligned}\right\}$$

$$\left.\begin{aligned} -1 - 1 &= -2 \\ -1 + 1 &= 0 \end{aligned}\right\}$$. This latter binomial must evidently remain real, since the difference into which the imaginary factor was to have been drawn vanishes.

3dly. Let $(x^3 - 7x + 6)$ be given, whose roots are $(1, 2 - 3)$. The binomials derived from these have been before given, in the 2d Example to the first Section of this Article; and the cube they produce shewn to be $(\overline{24 - \tfrac{80}{3}\sqrt{-\tfrac{1}{3}}})$, the cube root of which cannot be extracted; it being from the quadratic surd, it involves, in truth, *not a cube*, but a *truncate sixth* power in a cubic shape: and when, to remove its equivocal state, it is made rational, shews itself to be properly the sixth power equation $(x^6 - 6x^3 + \tfrac{343}{27} = 0)$, as before demonstrated.

26. This is the common reduction of a cubic equation, to one of the sixth degree but in form a quadratic, obtained, by clearing of its quadratic surd, the pure cubic formed by either of the two sets of binomials before described; and this is the only reduction of it yet discovered. Perhaps the method called CARDAN's rule, is the shortest mode of effecting this reduction; but I am not aware, that the *real principle* upon which it is founded has been any where fully analysed and explained, except in the foregoing investigation of it. The ordinary expositions of it certainly disclose nothing of the principle, and are even in many respects faulty; for they treat it as the effect of a supposition or lucky conjecture, when, in fact, there is no supposition or conjecture made; a regular clue, furnished by certain demonstrable peculiarities in some functions of this order of quantities, being pursued, till such a relationship amongst the roots may be inferred, as may be converted into equality at some known period. They also fail to account for the most striking part of the result; the irreducibility happening uniformly in cases where it has been supposed least to be expected, *i. e.* when the roots are real; which they refer to a particular limitation

in one of the steps taken, when it is, in truth, of much deeper origin than any particular method, being the necessary consequence of the constitution of the cube power.

27. The result of these observations upon cubic equations shews, that *directly* they are not resolvable, *i. e.* they cannot, like quadratics, be always brought to a mere extraction of their correspondent root: that, however, by means of the peculiarities inseparable from the number of *three quantities,* a relation is discoverable, which inevitably gives equal roots to the equation of the cubes of a particular function of them; but that, that function involves sometimes a quadratic surd which was not in the roots themselves, but arose from the form necessary to be given them; that the equal relation not taking place in any case, till the cube of that function, and, in some cases, not being *rational,* till the square of that cube, the equation is not lowered in degree, by the operation, but rather increased.

28. Let $(n = 4)$, and the equation become the general biquadratic $(x^4 - px^3 + qx^2 - rx + s = 0)$, the number of differences are twelve; we cannot, therefore, hope to obtain a direct simple resolution. But, in Art. 14, two peculiarities belonging to sets of four quantities were pointed out, from which it is easy to obtain a reduction of the equation to a cubic form. The *first* peculiarity there mentioned, was shewn to subsist among the sums of the combinations of the roots in pairs. If $(a, b, c, d,)$ be supposed the roots of the given equation, and their combinations by two $(ab, ac, ad, bc, bd, cd,)$ be summed in pairs, though the number of quantities so formed are no fewer than 30, yet there is an evident distinction observable amongst them; for, in some, (the first six,) *no* letter occurs twice. If, therefore, instead of simply requiring the

sums of the combinations of the roots in pairs, that function of the roots had been required, consisting of the sums of these combinations, *into the forming of which no root enters twice*, only six out of the whole number of combinations of the kind would answer that condition; and those six would be the same *three* repeated, for $(ab + cd$, and $cd + ab$ *&c.*) are the same quantities. So that the three quantities $(ab + cd, ac + bd, ad + bc,)$ would be the functions required, and all of the kind that can be made. Now, there is *no proposition* in the theory of equations more certain, than that the equation of any regular function of the roots may always be found by means of the known values of the coefficients.* As there are but three functions in this case, the resulting equation must consequently be a cubic; and, by taking the several combinations of the quantities $(ab + cd, ac + bd, ad + bc,)$, we may obtain their equation,

viz. $\underline{(x^3 - qx^2 + \overline{pr - 4s}\,x - p^2 s + 4qs - r^2 = 0)}$.

Therefore, *the finding the equation of that function of the roots of a biquadratic which arises from its combinations by two summed in pairs, so however that no root shall occur twice in any such sum, reduces the biquadratic to a cubic.*

29. Another peculiarity of four quantities is also given in Art. 14, *i. e.* that if taken originally so as to have their sum equal to nothing, the six quantities formed from their sums in pairs, will be *the same three* quantities taken both affirmatively and negatively. Then we know, by the reasoning in Art. 11, *the equation of those quantities (though of the sixth degree) will want every alternate term, or be of a cubic form;* accordingly, the equation of the function of the roots formed by sum-

* WARING'S *Med. Algeb.* cap. i. p. 1. *et infra.*

ming them in pairs, is $\overline{x-\frac{p}{2}}|^6 - \overline{2q+\frac{3}{4}p^2}| \times \overline{x-\frac{p}{2}}|^4 + \overline{q^2-4s-qp^2+pr+\frac{3}{16}p^4}| \times \overline{x-\frac{p}{2}}|^2 - \overline{\frac{p^3}{8}-\frac{p^4}{2}+r}|^2 = 0$,* which, when (p) is supposed to vanish, becomes $(x^6-2qx^4+\overline{q^2-4s}|x^2-r^2=0)$.

30. These two methods, *one* applying to the biquadratic equations complete in their terms, and *the other* to those from which the second term has been expunged, are all that have yet been discovered; and, notwithstanding the number of different methods attributed to different writers, which from their manner of setting out appear distinct, they will all be found to resolve themselves, *in principle,* into one of these. Dr. HUTTON's Mathematical Dictionary, under the article Biquadratic Equations, gives four methods; *viz.* FERRARI's, DES CARTES's, EULER's, and SIMPSON's; to which may be added another by Dr. WARING,† and perhaps many more. They proceed upon a variety of different contrivances; but, when analysed, and the real object gained is viewed apart from the process that led to it, FERRARI's, which is the oldest, *and does not require the extinction of the second term,* will be seen to produce the cubic $(x^3-qx^2+\overline{pr-4s}|x-p^2s+4qs-r^2=0)$; and DES CARTES's, *which supposes the second term to be first destroyed,* terminates in the cubic-formed equation of the sixth degree $(x^6-2qx^4+\overline{q^2-4s}|x^2-r^2=0)$. The rest produce cubics, or cubic-formed sixth powers, whose roots are some parts or multiples of this last; except WARING's method, *which does not expunge the second term,* and *therefore* produces a cubic whose roots are parts of the first. But, whether the resulting equation be

* WARING's *Medit. Algeb.* p. 133.

† Ibid. p. 138; and the Appendix to Dr. HUTTON's Dictionary.

that of the function, formed by summing the combinations by two of the roots in pairs, or summing the roots *themselves* in pairs, or the equation of the halves, or quarters, or doubles, trebles, *&c.* of those functions, is immaterial; *no new function* is employed, *no other principle* put in action, than what is derived from the general properties of this degree of quantities here explained.

31. Biquadratics being generally thus reducible to cubics, of course, by resolving those cubics, distinguishing what function their roots are of the roots of the original biquadratic, they may *all* be found; and, for practical utility, there is no preference to be made of either of the two methods; for, the first, though a *real cubic,* being formed from *products* of the roots, it requires a quadratic equation to obtain them after the cubic is resolved; whereas the *second,* though an equation of the sixth power, being formed from simple addition of the roots, gives them at once. But, as both these cubics necessarily have all their roots real, when those of the given biquadratic are so, and the resolution of cubics is in that case imaginary, it follows, that *no biquadratic having all its roots real,* can admit of a real solution by *either* of these methods.

32. The formula expressing the actual resolution of a biquadratic has not been given; the writers upon algebra going no further than to point out the cubics by means of which such a resolution may be obtained. To be sure, such a formula would be very long, and (till the imperfection in the cubic resolution, which must make a large part of it, can be removed,) embarrassed with radicals, so as to be of little practical use; but it would be a valuable accession to the theoretical part of algebra, to have the analysis of this degree carried as far as that of the

preceding, by developing every part of the functions that enter into the resolution, so as to be able to compose it at once, or make a complete reduction of the equation, without the intervention of any other steps.

33. Let (n) be taken $= (5)$, or any higher number. *Here*, the number of differences is increased to twenty; and, the higher we go the more they increase, so that a *direct simple resolution* is out of the question. Nor are we yet acquainted with any peculiarity attending five, or any higher number of quantities, upon which we can ground a relation to effect a reduction of any sort; wherefore, no method is known for equations of this and the higher orders. Whether any may *ever* be discovered, it is not easy to pronounce: if the reasoning from Art. 8 to Art. 15, of this Paper, be correct, there can be no chance, until some peculiar property of quantities of this class can be hit upon. It is perhaps a discouraging presumption against the existence of any such property, that no art nor labour has hitherto afforded the least clue to lead to one; but, on the other hand, it is impossible to tell what general properties of quantity may remain to be discovered; and, from the great distance the peculiarities of the degrees we have treated of lie from the surface, and their total want of order or connection with each other, it may be justly expected those of the higher degrees may lie still more detached and remote, beyond any efforts that have yet been made upon the subject. The proper method to proceed seems therefore to be, *abandoning all projects for the general resolution of equations, to investigate regularly the abstract properties of each separate order or number of quantities, turning them into all shapes, sifting all their combinations, and constructing and examining the equations of different complex functions of them, in order to see*

if latent peculiarities be not to be traced out in some of them. Wherever any distinguishing property is found, it will, by the principles here explained, infallibly lead to some method for the degree to which it belongs; and, whoever may be fortunate enough to discover any such property, in five, six, or any higher order of quantities, will have the honour of removing the important and hitherto impenetrable barrier, which has so long obstructed the farther improvement of algebra.